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# Resistance calculation of three-dimensional triangular and hexagonal prism lattices

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**Abstract.** The resistance between two arbitrary lattice sites in infinite three-dimensional triangular and hexagonal prism lattice networks of equal resistances, that have not been studied before, is computed by using lattice Green's function technique. For large separation between lattice points we numerically calculate the asymptotic value of the resistance for these lattices.

## 1 Introduction

One of the old traditional analysis methods for computing the resistance is Kirchhoff's laws [1], which can be applied in principle to any electric network of resistors, but with increasing the size of the network the problem becomes challenging task. In last two decades the problem of calculating the resistance of infinite resistor networks attracts a vast amount of attention in physics, mathematics, and electric engineering literatures as well. Various techniques have been employed by researchers to study this problem [2–6]. These methods basically involve difference equations governed by Ohm's and Kirchhoff's laws.

In 1999, Atkinson *et al.* [4] used complex Fourier transforms and generalized the method of Venezian [5] to cubic and hypercubic lattices in higher dimensions, as well as to triangular and hexagonal (honeycomb) lattices in two dimensions. In 2000 Cserti [6] presented a method based on the lattice Green's function to compute the resistance for infinite $d$-dimensional hypercubic, rectangular, triangular, and honeycomb lattices of resistors. For the first time Cserti obtained the recurrence relations for the resistance of an infinite square lattice using the recurrence formulas for the Green's functions derived by Morita [7]. Two years later, Cserti *et al.* [8] established a method based on the lattice Green's function to compute the resistance of the perturbed network in which one of the resistors is missing in the perfect lattice has only one basis.

In 2011 Cserti *et al.* [9] generalized the lattice Green's function method [6] for calculating the equivalent resistance of any infinite periodic lattice structure of resistor networks. Following the approach of refs. [6,8,9], several considerable works have been presented in the literature [10–26]. In [20], one of us generalized the Green's function method developed in [8] to find the two-point resistance on the perturbed uniform tiling in which each unit cell has any number of lattice sites.

In the case of finite networks, two main methods were established to compute the two-point resistances. The first is usually called Laplacian approach which is based on finding eigenvalues and eigenvectors of the Laplacian matrix [27]. The Laplacian method was modified for different types of resistor networks [28–30]. The second method is called the Recursion-Transform method [31], and it is based on the solution of a recurrence relation found by a matrix transformation of the equations involving the column currents. This method was developed recently to resolve many resistor networks of various topologies [32–37].

In [38], an explicit formula for two-point resistance in non-symmetric finite networks in terms of the eigenvalues and mutually orthogonal basis of left- and right-hand eigenvectors of the Laplacian matrix also appeared in the literature.

In the present work, we apply the lattice Green's function method of refs. [6,9] to infinite resistor prism lattices in three dimensions (see figs. 1 and 4); namely, we evaluate the two-point resistance on the triangular prism and hexagonal prism lattices of equal electrical resistors (to the best our knowledge these lattices provide a wealth of examples of

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new resistor networks not considered in the literature). Here, throughout this paper, we think it is convenient to use the orthogonal Cartesian coordinates system [39] instead of a triangle coordinates system for several reasons such as:

i) It is easier to follow for two-dimensional triangular lattice compared to other coordinate systems.
ii) The integrals we obtained are more suitable for numerical calculations by using Mathematica.
iii) Recurrence formulas for resistances can be derived, but we did not present them in the paper.

The paper is organized as follows: we begin with a brief review of the general formulation [9] to determine the resistance between two arbitrary lattice points in an infinite periodic lattice of any resistor network (sect. 2). In sect. 3, we study the resistance for an infinite $3D$ triangular prism lattice. In sect. 4, we present the types of the resistances for an infinite $3D$ hexagonal prism lattice.

## 2 Formulation of the two-point resistance

In this section, we briefly review the formulation of two-point resistance on a periodic lattice structure of resistor networks. A detailed formulation can be found in the ref. [9].

Consider an infinite lattice structure that is a uniform tiling of $d$-dimensional space with identical resistances $R$. The lattice points can be represented by the vector $\boldsymbol{r} = \ell_1 \boldsymbol{a}_1 + \ell_2 \boldsymbol{a}_2 + \cdots + \ell_d \boldsymbol{a}_d$, where $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_d$ are the unit cell vectors in the $d$-dimensional space and $\ell_1, \ell_2, \cdots, \ell_d$ are arbitrary integers. If the unit cell contains $s$ lattice points labeled by $\alpha = 1, 2, \cdots, s$, then denote by $\{\boldsymbol{r}; \alpha\}$ any lattice point, where $\boldsymbol{r}$ and $\alpha$ specify the unit cell and the lattice point, and let $U_\alpha(\boldsymbol{r})$ and $I_\alpha(\boldsymbol{r})$ be the electric potential and current at point $\{\boldsymbol{r}; \alpha\}$, respectively.

According to Kirchhoff's current and Ohm's laws, the current $I_\nu(\boldsymbol{r})$ entering, from a source outside lattice, the lattice point $\{\boldsymbol{r}; \nu\}$ in the unit cell can be written as

$$\sum_{\boldsymbol{r}', \beta} L_{\nu\beta}(\boldsymbol{r}, \boldsymbol{r}') U_\beta(\boldsymbol{r}') = -R I_\nu(\boldsymbol{r}), \tag{1}$$

where $L_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}')$ is a $s$ by $s$ usually called Laplacian matrix of the lattice.

To calculate the resistance $R_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ between two lattice points $\{\boldsymbol{r}_1; \alpha\}$ and $\{\boldsymbol{r}_2; \beta\}$, one connects these points to the two terminals of an external source and measure the current going through the source while no other lattice points are connected to external sources. Then, the resistance $R_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is given by Ohm's law:

$$R_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{U_\alpha(\boldsymbol{r}_1) - U_\beta(\boldsymbol{r}_2)}{I}. \tag{2}$$

The computation of the two-point resistance $R_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is now reduced to solving eq. (1) for $U_\alpha(\boldsymbol{r}_1)$ and $U_\beta(\boldsymbol{r}_2)$ by using the lattice Green's function with the current distribution given by

$$I_\nu(\boldsymbol{r}) = I \left(\delta_{\boldsymbol{r}, \boldsymbol{r}_1} \delta_{\alpha,\nu} - \delta_{\boldsymbol{r}, \boldsymbol{r}_2} \delta_{\beta,\nu}\right). \tag{3}$$

The lattice Green's function is formally defined as

$$G = -L^{-1}. \tag{4}$$

Hence, eq. (1) can be written as

$$U_\mu(\boldsymbol{r}') = R \sum_{\nu, \boldsymbol{r}} G_{\mu\nu}(\boldsymbol{r}', \boldsymbol{r}) I_\nu(\boldsymbol{r}), \tag{5}$$

where $G_{\alpha\beta}(\boldsymbol{r}', \boldsymbol{r}) = G_{\alpha\beta}(\boldsymbol{r}' - \boldsymbol{r})$, it is a $s$ by $s$ matrix. Substituting eq. (3) into (5), one obtains

$$U_\mu(\boldsymbol{r}') = RI \left[G_{\mu\alpha}(\boldsymbol{r}', \boldsymbol{r}_1) - G_{\mu\beta}(\boldsymbol{r}', \boldsymbol{r}_2)\right]. \tag{6}$$

Using eq. (6) in (2) the two-point resistance in terms of lattice Green's functions can be obtained as

$$R_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = RI \left[G_{\alpha\alpha}(\boldsymbol{r}_1, \boldsymbol{r}_1) + G_{\beta\beta}(\boldsymbol{r}_2, \boldsymbol{r}_2) - G_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) - G_{\beta\alpha}(\boldsymbol{r}_2, \boldsymbol{r}_1)\right]. \tag{7}$$

Now the lattice Green's function $G_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ can be given by its Fourier transform $G_{\alpha\beta}(\boldsymbol{k})$ as

$$G_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{V_c}{(2\pi)^d} \int_{BZ} \mathrm{d}\boldsymbol{k} G_{\alpha\beta}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}_2 - \boldsymbol{r}_1)}, \tag{8}$$

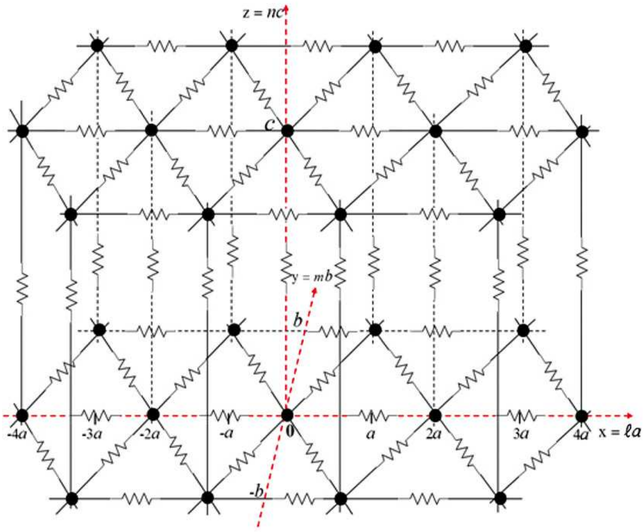


**Fig. 1.** The resistor network of the three-dimensional triangular prism lattice.

where $V_c$ is the volume of the unit cell and $\boldsymbol{k} = (k_1, k_2, \cdots k_d)$ is the wave vector in the $d$-dimensional Fourier space (in the reciprocal lattice) and is limited to the first Brillouin zone (BZ) which is a $d$-dimensional hypercube with sides $k_1 = 2\pi/a_1, k_2 = 2\pi/a_2 \cdots k_d = 2\pi/a_d$. Thus, eq. (7) can be written as

$$R_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{RV_c}{(2\pi)^d} RI \left[ G_{\alpha\alpha}(\boldsymbol{k}) + G_{\beta\beta}(\boldsymbol{k}) - G_{\alpha\beta}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)} - G_{\beta\alpha}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)} \right]. \tag{9}$$

By writing $\boldsymbol{r}_2 - \boldsymbol{r}_1 = \ell_1 \boldsymbol{a}_1 + \ell_2 \boldsymbol{a}_2 + \cdots + \ell_d \boldsymbol{a}_d$ and changing the variables $\boldsymbol{k} \cdot \boldsymbol{a}_i$ $(i = 1, 2, \cdots, d)$ with $V_c = a_1 a_2 \cdots a_d$, eqs. (8) and (9) can be simplified to

$$G_{\alpha\beta}(\ell_1, \cdots, \ell_d) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_d}{2\pi} G_{\alpha\beta}(\theta_1, \cdots, \theta_d) e^{-i(\ell_1\theta_1 + \cdots + \ell_d\theta_d)}, \tag{10}$$

$$\begin{aligned} R_{\alpha\beta}(\ell_1, \cdots, \ell_d) = R \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_d}{2\pi} \Big\{ & G_{\alpha\alpha}(\theta_1, \cdots, \theta_d) + G_{\beta\beta}(\theta_1, \cdots, \theta_d) \\ & - G_{\alpha\beta}(\theta_1, \cdots, \theta_d) e^{-i(\ell_1\theta_1 + \cdots + \ell_d\theta_d)} - G_{\beta\alpha}(\theta_1, \cdots, \theta_d) e^{i(l_1\theta_1 + \cdots + \ell_d\theta_d)} \Big\}. \end{aligned} \tag{11}$$

If the unit cell contains only one lattice point (*i.e.* $s = 1$ and $\alpha = \beta = 1$), then $L_{\alpha\beta}$ and $G_{\alpha\beta}$ are the lattice Laplacian and the lattice Green's function ($1 \times 1$ matrices) corresponding to the finite-difference representation of the Laplace operator [6]. In this case the lattice Green's function and the resistance can be written as [6].

$$G(\ell_1, \cdots, \ell_d) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_d}{2\pi} G(\theta_1, \cdots, \theta_d) e^{-i(\ell_1\theta_1 + \cdots + \ell_d\theta_d)}, \tag{12}$$

$$R(\ell_1, \ell_2, \cdots, \ell_d) = 2R \left[ G(0, 0, \cdots, 0) - G(\ell_1, \ell_2, \cdots, \ell_d) \right]. \tag{13}$$

## 3 Three-dimensional triangular prism lattice network

Consider an infinite three-dimensional triangular prism lattice network consisting of identical resistors $R$ as shown in fig. 1. The bases of the prism are two-dimensional triangular lattices (see fig. 2). Let $\boldsymbol{r} = \ell\boldsymbol{a} + m\boldsymbol{b} + n\boldsymbol{c}$ be the lattice site: $\ell a$, $mb$ and $nc$ are (orthogonal coordinate axes) along the $x$, $y$ and $z$-axis, respectively, where $\ell + m$ is an even integer and $n$ is any integer. If the distance between adjacent nodes on two-dimensional triangular lattices is chosen to be equal 1, then $a$ and $b$ are $1/2$ and $\sqrt{3}/2$, respectively. The unit cell contains only one lattice point and each lattice point has eight neighbors: $\pm\boldsymbol{2a}$, $\pm\boldsymbol{a} \pm \boldsymbol{b}$, $\pm\boldsymbol{c}$.

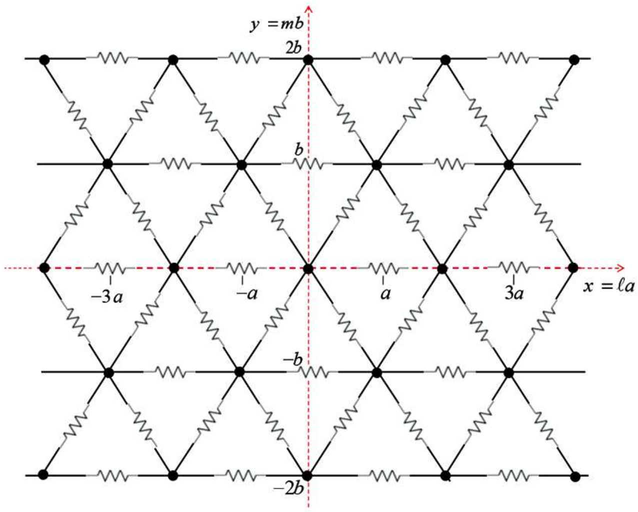


**Fig. 2.** The basal plane (triangular lattices) of the triangular prism lattice with the two perpendicular coordinates $\ell a$ and $mb$.

Following the same way as in [9] we calculate the two-point resistance on the triangular prism resistor network. By a combination of Kirchhoff's current rule and Ohm's law the current $I(\boldsymbol{r})$ at lattice site $\boldsymbol{r}$ can be written as

$$
\begin{aligned}
I(\boldsymbol{r}) = {} & \frac{U(\boldsymbol{r}) - U(\boldsymbol{r}+\boldsymbol{2a})}{R} + \frac{U(\boldsymbol{r}) - U(\boldsymbol{r}-\boldsymbol{2a})}{R} + \frac{U(\boldsymbol{r}) - U(\boldsymbol{r}+\boldsymbol{a}+\boldsymbol{b})}{R} + \frac{U(\boldsymbol{r}) - U(\boldsymbol{r}-\boldsymbol{a}-\boldsymbol{b})}{R} \\
& + \frac{U(\boldsymbol{r}) - U(\boldsymbol{r}+\boldsymbol{a}-\boldsymbol{b})}{R} + \frac{U(\boldsymbol{r}) - U(\boldsymbol{r}-\boldsymbol{a}+\boldsymbol{b})}{R} + \frac{U(\boldsymbol{r}) - U(\boldsymbol{r}+\boldsymbol{c})}{R} + \frac{U(\boldsymbol{r}) - U(\boldsymbol{r}-\boldsymbol{c})}{R}.
\end{aligned} \tag{14}
$$

The electric potential and current at any site are usually written by their Fourier transforms:

$$
U(\boldsymbol{r}) = \frac{V_c}{(2\pi)^3}\int_{BZ} \mathrm{d}\boldsymbol{k}U(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{15}
$$

$$
I(\boldsymbol{r}) = \frac{V_c}{(2\pi)^3}\int_{BZ} \mathrm{d}\boldsymbol{k}I(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{16}
$$

where $V_c = abc$ is the volume of the unit cell and $\boldsymbol{k} = (k_x, k_y, k_z)$ is the wave vector in the three-dimensional Fourier space and is limited to the first Brillouin zone which is a rectangular box with sides $k_x = 2\pi/a$, $k_y = 2\pi/b$, $k_z = 2\pi/c$. Substituting eqs. (15) and (16) into (14) we have

$$
L(\boldsymbol{k})U(\boldsymbol{k}) = -RI(\boldsymbol{k}), \tag{17}
$$

where $L(\boldsymbol{k})$ is the Fourier transform of the Laplacian operator $L(\boldsymbol{r})$ of the triangular prism lattice, given by

$$
L(\boldsymbol{k}) = -8 + 2\cos 2\boldsymbol{k}\cdot\boldsymbol{a} + 4\cos\boldsymbol{k}\cdot\boldsymbol{a}\cos\boldsymbol{k}\cdot\boldsymbol{b} + 2\cos\boldsymbol{k}\cdot\boldsymbol{c}. \tag{18}
$$

Inverting $L(\boldsymbol{k})$ the Fourier transform of the lattice Green's function $G(\boldsymbol{k})$ is

$$
G(\boldsymbol{k}) = -L^{-1}(\boldsymbol{k}) = \frac{1}{8 - 2\cos 2\boldsymbol{k}\cdot\boldsymbol{a} - 4\cos\boldsymbol{k}\cdot\boldsymbol{a}\cos\boldsymbol{k}\cdot\boldsymbol{b} - 2\cos\boldsymbol{k}\cdot\boldsymbol{c}}. \tag{19}
$$

The lattice Green's function $G(\boldsymbol{r})$ can be given by its Fourier transform as

$$
G(\boldsymbol{r}) = \frac{V_c}{(2\pi)^3}\int_{BZ} \mathrm{d}\boldsymbol{k}G(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{20}
$$

Equation (20) can written as

$$
G(\ell a, mb, nc) = \frac{abc}{(2\pi)^3}\int_{-\pi/a}^{\pi/a}\mathrm{d}k_x\int_{-\pi/b}^{\pi/b}\mathrm{d}k_y\int_{-\pi/c}^{\pi/c}\mathrm{d}k_z\frac{e^{i(\ell k_x + mk_y + nk_z)}}{8 - 2\cos 2\boldsymbol{k}\cdot\boldsymbol{a} - 4\cos\boldsymbol{k}\cdot\boldsymbol{a}\cos\boldsymbol{k}\cdot\boldsymbol{b} - 2\cos\boldsymbol{k}\cdot\boldsymbol{c}}. \tag{21}
$$

**Table 1.** Numerical values of the resistance $R(\ell, m, n)$ in units of $R$ in a triangular prism lattice.

| $\ell$, $m$, $n$ | $R(\ell, m, n)/R$ | $\ell$, $m$, $n$ | $R(\ell, m, n)/R$ |
|---|---|---|---|
| $0, 0, 0$ | $0, 0, 0$ | $50, 0, 0$ | 0.357822 |
| $0, 0, 1$ | 0.259513 | $0, 50, 0$ | 0.359726 |
| $1, 1, 0$ | 0.246829 | $0, 0, 50$ | 0.360487 |
| $2, 0, 0$ | 0.246829 | $100, 0, 0$ | 0.360074 |
| $1, 1, 1$ | 0.293775 | $200, 0, 0$ | 0.361199 |
| $2, 0, 1$ | 0.293775 | $500, 0, 0$ | 0.361875 |
| $4, 4, 1$ | 0.335369 | $1000, 0, 0$ | 0.36210 |
| $5, 5, 0$ | 0.339645 | $0, 1000, 0$ | 0.362195 |
| $10, 0, 0$ | 0.339645 | $0, 0, 1000$ | 0.362233 |
| $10, 10, 10$ | 0.355211 | $10000, 0, 0$ | 0.362303 |
| $30, 0, 10$ | 0.356516 | $\infty$ | 0.362325 |

By changing the variables $k_x = \theta_x/a$, $k_y = \theta_y/b$ and $k_z = \theta_z/c$, eq. (21)can be simplified to

$$G(\ell, m, n) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_z}{2\pi} \frac{\cos \ell\theta_x \cos m\theta_y \cos n\theta_z}{8 - 2\cos 2\theta_x - 4\cos\theta_x \cos\theta_y - 2\cos\theta_z}. \tag{22}$$

The equivalent resistance between the origin and lattice point $(\ell, m, n)$ in triangular prism lattice is given by eq. (13)

$$R(\ell, m, n) = 2R\,[G(0, 0, 0) - G(\ell, m, n)] \tag{23}$$

and substituting eq. (22) into (23) we have

$$R(\ell, m, n) = R \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_z}{2\pi} \frac{1 - \cos \ell\theta_x \cos m\theta_y \cos n\theta_z}{4 - \cos 2\theta_x - 2\cos\theta_x \cos\theta_y - \cos\theta_z}. \tag{24}$$

Using the expression in result (24), one can calculate the resistance between the origin and lattice point $(\ell, m, n)$. As an example, the resistance between the origin and point $(1, 1, 0)$ is given by

$$R(1, 1, 0) = R \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_z}{2\pi} \frac{1 - \cos\theta_x \cos\theta_y}{4 - \cos 2\theta_x - 2\cos\theta_x \cos\theta_y - \cos\theta_z}. \tag{25}$$

Performing the integration over the variable $\theta_z$ by residue theorem and evaluating the others numerically using Mathematica, we find that $R(1, 1, 0) \approx 0.246829R$. As another example, the resistance between the origin and point $(0, 0, 1)$ is

$$R(0, 0, 1) = R \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_z}{2\pi} \frac{1 - \cos\theta_z}{4 - \cos 2\theta_x - 2\cos\theta_x \cos\theta_y - \cos\theta_z}, \tag{26}$$

which gives to 0.259513. Numerical values for the resistance between the origin and additional points are listed in table 1.

It is interesting to note that the resistance tends to a finite value as the separation between lattice sites goes to infinity. This can be shown from Riemann-Lebesque Lemma which states: If $\boldsymbol{f}(t)$ is an integrable function on the interval $[a, b]$, then

$$\lim_{p \longrightarrow \infty} \int_a^b \boldsymbol{f}(t) \cos pt\mathrm{d}t. \tag{27}$$

Hence, from eq. (23) $G(\ell, m, n) \longrightarrow 0$ and thus, using eq. (24) the asymptotic value of the resistance is

$$R(\ell, m, n) \longrightarrow 2RG(0, 0, 0), \tag{28}$$

as any of the $\ell, m, n \longrightarrow \infty$. The numerical asymptotic value of the resistance can be calculated to be

$$2RG(0, 0, 0) = 0.362325 \cdots R. \tag{29}$$

In fig. 3, the numerical values of the resistance in units of $R$ is plotted *versus* along the $z$-axis.

From both table 1 and fig. 3 one can see obviously that the resistance approaches rapidly to its asymptotic value given above.

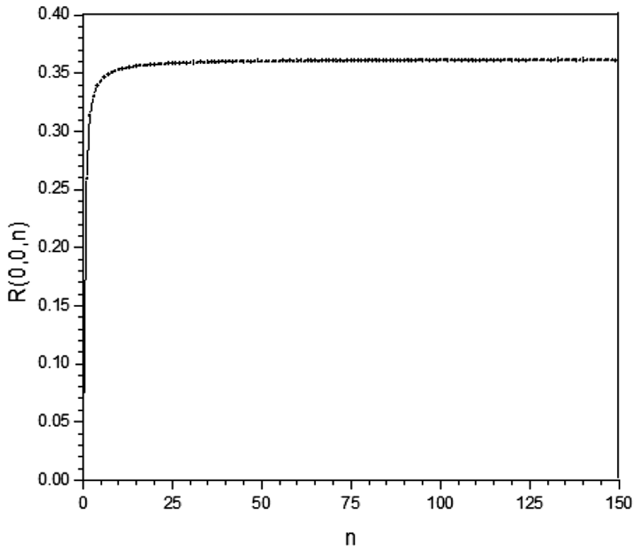


**Fig. 3.** The resistance of units of $R$ along the $nc$ axis for a triangular prism lattice network.

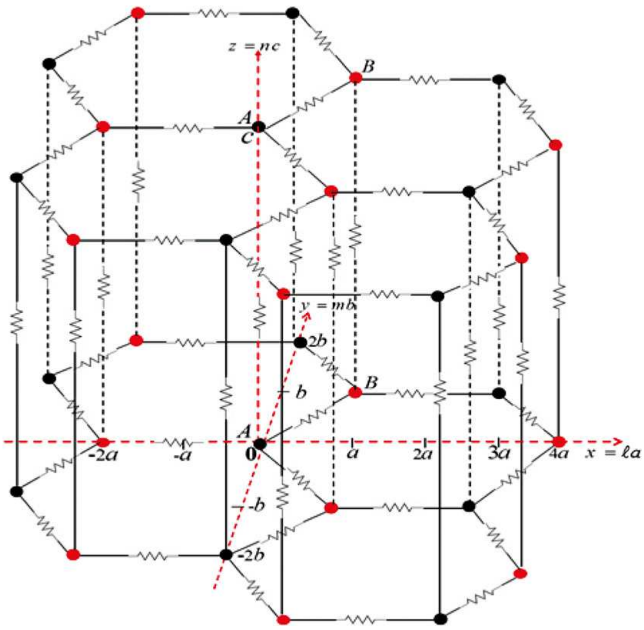


**Fig. 4.** The resistor network of the three-dimensional hexagonal prism lattice.

## 4 Three-dimensional hexagonal prism lattice network

In this section, we follow ref. [9] to calculate the two-point resistance in an infinite three-dimensional regular hexagonal prism lattice. Consider a three-dimensional hexagonal prism lattice of equal resistances $R$ as shown in fig. 4. The bases are two-dimensional regular hexagonal (honeycomb) lattices (see fig. 5). The unit cell contains two lattice points labeled by $\alpha = A, B$. Again let the lattice vector $\boldsymbol{r} = \ell\boldsymbol{a} + m\boldsymbol{b} + n\boldsymbol{c}$ specifies the lattice points, where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are the orthogonal unit cell vectors along the $x$, $y$ and $z$-axis, respectively, $\ell + m$ is an even integer and $n$ is any integer. If the distance between adjacent nodes on two-dimensional honeycomb lattices is chosen to be equal 2, then $a$ and $b$ are 1 and $\sqrt{3}$, respectively.

Using Kirchhoff's current law and then Ohm's law, the currents at lattice points $\{\boldsymbol{r}; A\}$ and $\{\boldsymbol{r}; B\}$ in the unit cell can be written as the finite difference equations:

$$\begin{aligned} I_A(\boldsymbol{r}) = {} & \frac{U_A(\boldsymbol{r}) - U_B(\boldsymbol{r} - \boldsymbol{2a})}{R} + \frac{U_A(\boldsymbol{r}) - U_B(\boldsymbol{r} + \boldsymbol{a} + \boldsymbol{b})}{R} + \frac{U_A(\boldsymbol{r}) - U_B(\boldsymbol{r} + \boldsymbol{a} - \boldsymbol{b})}{R} \\ & + \frac{U_A(\boldsymbol{r}) - U_A(\boldsymbol{r} + \boldsymbol{c})}{R} + \frac{U_A(\boldsymbol{r}) - U_A(\boldsymbol{r} - \boldsymbol{c})}{R}, \end{aligned} \tag{30}$$

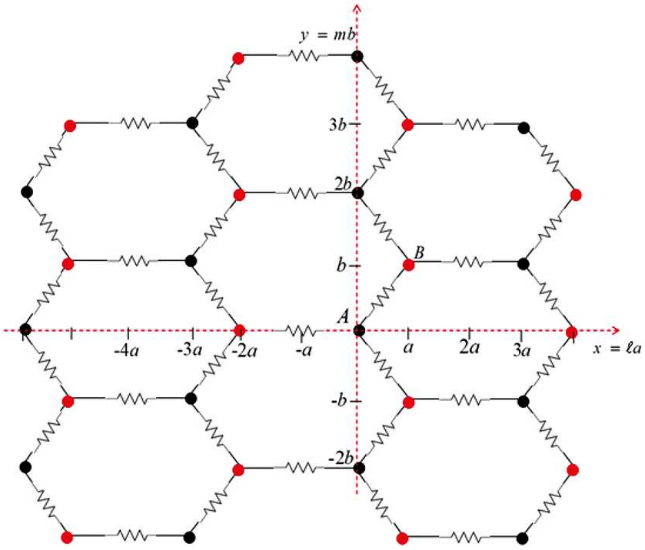


**Fig. 5.** The basal plane (hexagonal lattices) of the hexagonal prism lattice with the two perpendicular coordinates $\ell a$ and $mb$.

$$I_B(\boldsymbol{r}) = \frac{U_B(\boldsymbol{r}) - U_A(\boldsymbol{r}+\boldsymbol{2a})}{R} + \frac{U_B(\boldsymbol{r}) - U_A(\boldsymbol{r}-\boldsymbol{a}-\boldsymbol{b})}{R} + \frac{U_B(\boldsymbol{r}) - U_A(\boldsymbol{r}-\boldsymbol{a}+\boldsymbol{b})}{R} + \frac{U_B(\boldsymbol{r}) - U_B(\boldsymbol{r}+\boldsymbol{c})}{R} + \frac{U_B(\boldsymbol{r}) - U_B(\boldsymbol{r}-\boldsymbol{c})}{R}. \tag{31}$$

The electric potential and current can be given by their Fourier transforms:

$$U_\alpha(\boldsymbol{r}) = \frac{V_c}{(2\pi)^3}\int_{BZ} \mathrm{d}\boldsymbol{k} U_\alpha(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{32}$$

$$I_\alpha(\boldsymbol{r}) = \frac{V_c}{(2\pi)^3}\int_{BZ} \mathrm{d}\boldsymbol{k} I_\alpha(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{33}$$

where $\alpha = A, B$. Using eqs. (32) and (33) into (30) and (31), we have

$$\boldsymbol{L}(\boldsymbol{k})\begin{bmatrix} U_A(\boldsymbol{k}) \\ U_B(\boldsymbol{k}) \end{bmatrix} = -R\begin{bmatrix} I_A(\boldsymbol{k}) \\ I_B(\boldsymbol{k}) \end{bmatrix}, \tag{34}$$

where $\boldsymbol{L}(\boldsymbol{k})$ ($2\times 2$ matrix) is the Fourier transform the Laplacian matrix of the hexagonal prism lattice, given by

$$\boldsymbol{L}(\boldsymbol{k}) = \begin{bmatrix} -5 + 2\cos\boldsymbol{k}\cdot\boldsymbol{c} & e^{-2i\boldsymbol{k}\cdot\boldsymbol{a}} + 2e^{i\boldsymbol{k}\cdot\boldsymbol{a}}\cos\boldsymbol{k}\cdot\boldsymbol{b} \\ e^{2i\boldsymbol{k}\cdot\boldsymbol{a}} + 2e^{-i\boldsymbol{k}\cdot\boldsymbol{a}}\cos\boldsymbol{k}\cdot\boldsymbol{b} & -5 + 2\cos\boldsymbol{k}\cdot\boldsymbol{c} \end{bmatrix}. \tag{35}$$

Inverting the matrix $\boldsymbol{L}(\boldsymbol{k})$and changing the variables $\theta_x$, $\theta_y$ and $\theta_z$ in eq. (35) by $\boldsymbol{k}\cdot\boldsymbol{a}$, $\boldsymbol{k}\cdot\boldsymbol{b}$ and $\boldsymbol{k}\cdot\boldsymbol{c}$, respectively, the lattice Green's function becomes

$$\boldsymbol{G}(\theta_x,\theta_y,\theta_z) = \frac{1}{\det\boldsymbol{L}}\begin{bmatrix} 5 - 2\cos\theta_z & e^{-2i\theta_x} + 2e^{i\theta_x}\cos\theta_y \\ e^{2i\theta_x} + 2e^{-i\theta_x}\cos\theta_y & 5 - 2\cos\theta_z \end{bmatrix}, \tag{36}$$

where $\det\boldsymbol{L} = 2(12 - 2\cos 3\theta_x\cos\theta_y - \cos 2\theta_y - 10\cos\theta_z + \cos 2\theta_z)$ is the determinant of the matrix $\boldsymbol{L}(\theta_x,\theta_y,\theta_z)$.

The equivalent resistance $R_{\alpha\beta}(\ell,m,n)$ between the origin $\{\boldsymbol{0};\alpha = A,B\}$ and any node $\{(\ell,m,n);\beta = A,B\}$ can be calculated from eq. (11) for $d=3$,

$$R_{\alpha\beta}(\ell,m,n) = R\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_x}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_y}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_z}{2\pi}\Big\{G_{\alpha\alpha}(\theta_x,\theta_y,\theta_z) + G_{\beta\beta}(\theta_x,\theta_y,\theta_z) - G_{\alpha\beta}(\theta_x,\theta_y,\theta_z)e^{-i(\ell\theta_x+m\theta_y+n\theta_z)} - G_{\beta\alpha}(\theta_x,\theta_y,\theta_z)e^{i(\ell\theta_x+m\theta_y+n\theta_z)}\Big\}. \tag{37}$$

**Table 2.** Numerical values of the resistances $R_{AA}(\ell, m, n)$ and $R_{AB}(\ell, m, n)$ in units of $R$ in a hexagonal prism lattice.

| $\ell$, $m$, $n$ | $R_{AA}(\ell, m, n)/R$ | $\ell$, $m$, $n$ | $R_{AB}(\ell, m, n)/R$ |
|---|---|---|---|
| $0, 0, 1$ | 0.395256 | $-2, 0, 0$ | 0.403162 |
| $0, 0, 2$ | 0.519258 | $1, 1, 0$ | 0.403162 |
| $0, 2, 0$ | 0.522231 | $1, 3, 1$ | 0.578202 |
| $3, 1, 0$ | 0.522231 | $-2, 2, 0$ | 0.550135 |
| $0, 2, 1$ | 0.543021 | $1, 1, 1$ | 0.485675 |
| $3, 1, 1$ | 0.543021 | $4, 0, 0$ | 0.550136 |
| $3, 3, 1$ | 0.588895 | $4, 2, 1$ | 0.578202 |
| $6, 0, 0$ | 0.585211 | $1, 1, 3$ | 0.577664 |
| $6, 0, 4$ | 0.613063 | $1, 1, 5$ | 0.610659 |
| $9, 3, 0$ | 0.61886 | $7, 3, 2$ | 0.614015 |
| $6, 6, 6$ | 0.634946 | $10, 0, 0$ | 0.617366 |
| $0, 10, 0$ | 0.640337 | $7, 9, 0$ | 0.636991 |
| $12, 12, 0$ | 0.645078 | $4, 0, 10$ | 0.638054 |
| $0, 24, 0$ | 0.653493 | $100, 0, 0$ | 0.660206 |
| $0, 2, 100$ | 0.6622244 | $1, 1, 100$ | 0.6622243 |

There are four types of resistances:

$A$-$A$ type resistance: The resistance between lattice sites $\{\mathbf{0}; = A\}$ and $\{(\ell, m, n); = A\}$ is given by

$$R_{AA}(\ell, m, n) = R \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_z}{2\pi} \frac{(5 - 2\cos\theta_z)(1 - \cos\ell\theta_x \cos m\theta_y \cos n\theta_z)}{12 - 2\cos 3\theta_x \cos\theta_y - \cos 2\theta_y - 10\cos\theta_z + \cos 2\theta_z}. \tag{38}$$

Like in a simple hexagonal lattice, in a hexagonal prism lattice the two-point resistance $R_{AA}(\ell, m, n)$ tends to a finite value (*i.e.* asymptotic value of $2RG_{AA}(0, 0, 0) = 0.664978\ldots R$) as the distance between the lattice sites goes to infinity.

$A$-$B$ type resistance: The resistance between the lattice sites $\{\mathbf{0}; = A\}$ and $\{(\ell, m, n); = B\}$:

$$R_{AB}(\ell, m, n) = R \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_z}{2\pi}$$
$$\times \frac{\{5 - 2\cos\theta_z - \cos[(\ell+2)\theta_x + m\theta_y + n\theta_z] - 2\cos\theta_y \cos[(\ell-1)\theta_x + m\theta_y + n\theta_z]\}}{12 - 2\cos 3\theta_x \cos\theta_y - \cos 2\theta_y - 10\cos\theta_z + \cos 2\theta_z}. \tag{39}$$

From the lattice symmetric the other two types of the resistance are

$$R_{BB}(\ell, m, n) = R_{AA}(\ell, m, n), \quad \text{and} \quad R_{BA}(\ell, m, n) = R_{AB}(\ell, m, n). \tag{40}$$

In table 2 the numerical values of the resistances $R_{AA}(\ell, m, n)$ and $R_{AB}(\ell, m, n)$ were calculated by Mathematica.

It can also be observed, from table 2, that the resistances $R_{AA}(\ell, m, n)$ and $R_{AB}(\ell, m, n)$ are approximately equal and tend rapidly to the asymptotic value $(0.664978\ldots R)$ as the separation between lattice sites increases.

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